Prüfer-Like Conditions in Subring Retracts and Applications

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Abstract. In this paper, we consider five possible extensions of the Prüfer domain notion to the case of commutative rings with zero divisors. We investigate the transfer of these Prüfer-like properties between a commutative ring and its subring retract. Our results generate new families of examples of rings subject to a given Prüfer-like conditions.

Key Words. Prüfer rings, Gaussian rings, arithmetical rings, weak global dimension of rings, semihereditary rings, subring retract, trivial ring extensions, Nagata rings.

1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unital.

In his article [25], Prüfer introduced a new class of integral domains, namely those domains R in which all finitely generated ideals are invertible. Through the years, Prüfer domains acquired a great many equivalent characterizations, each of wich can, and was, extended to rings with zero-divisors in a number of ways. More precisely, we consider the following Prüfer-like properties on a commutative ring ([3] and [4]):

(1) R is semihereditary, i.e., every finitely generated ideal is projective.
(2) The weak global dimension of R is at most one.
(3) R is an arithmetical ring, i.e., every finitely generated ideal is locally principal.
(4) R is a Gaussian ring, i.e., $C_R(fg) = C_R(f)C_R(g)$ for any polynomials f, g with coefficients in R, where $C_R(f)$ is the ideal of R generated by the coefficients of f called the content ideal of f.
(5) R is a Prüfer ring, i.e., every finitely generated regular ideal is invertible (equivalently, every

two-generated regular ideal is invertible).

In [11], it is proved that each one of the above conditions implies the following next one (i.e., (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5)), and examples are given to show that in general the implications can't be reversed. Moreover, an investigation is carried out to see which conditions may be added to some of the preceding properties in order to reverse the implications.

Recall that in the domain context, the five classes of Prüfer-like rings collapse to the notion of Prüfer domain. From Bazzoni and Glaz [4, Theorem 3.12], we note that a Prüfer ring R satisfies one of the five conditions if and only if the total ring of quotients $Tot(R)$ of R satisfies the same condition. See for instance [3, 4, 7, 10, 11, 13, 17, 23, 28].

For two rings $A \subseteq B$, we say that A is a module retract (or a subring retract) of B if there exists an A-module homomorphism $\phi : B \longrightarrow A$ such that $\phi|_A = id|_A$; ϕ is called a module retraction map. If such a map ϕ exists, B contains A as an A-module direct summand.
Considerable works have been concerned with the descent and ascent of a variety of finiteness and related homological properties between a ring and its subring retract. See for instance [5, 6, 9, 18, 19, 24].

A special application of subring retract is the notion of trivial ring extension. Let A be a ring, E an A-module and $R = A \propto E$, the set of pairs (a, e) with $a \in A$ and $e \in E$, under coordinatewise addition and under an adjusted multiplication defined by $(a, e)(a', e') = (aa', ae' + a'e)$, for all $a, a' \in A, e, e' \in E$. Then R is called the trivial ring extension of A by E. It is clear that A is a module retract of R, where the module retraction map ϕ is defined by $\phi(x, e) = x$.
Trivial ring extensions have been studied extensively; the work is summarized in Glaz [8] and Huckaba [16]. These extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [8, 16, 20].

In this article we investigate the transfer of the Prüfer-like properties between a commutative ring and its subring retract. Our results generate new and original examples which enrichy the current literature with new families of Prüfer-like rings with zerodivisors.

2 Prüfer-like properties in subring retract

In this section we investigate the transfer of Gaussian, Prüfer, and arithmetical properties between a ring and its subring retract.
We begin by studying the transfer of Gaussian property. Recall that $Nil(R)$ is the set of nilpotent elements in a ring R.

Theorem 2.1 *Let R be a ring and A a subring retract of R.*
1) *If R is a Gaussian ring then so is A.*
2) *Assume that (A, M) is a local ring and $R := A \propto (A/M)$ be the trivial ring extension of A by A/M. Then R is a Gaussian ring if and only if so is A.*

Proof. 1) Assume that R is a Gaussian ring and let $f(X) = \sum_{i=0}^{n} a_i X^i$, $g(X) = \sum_{i=0}^{m} b_i X^i$ be two polynomials of $A[X]$, where n and m are two positive integers. Our aim is to prove that $C_A(f)C_A(g) \subseteq C_A(fg)$. Let α be an element of $C_A(f)C_A(g)$, we have $\alpha \in C_R(f)C_R(g)$ so, since R is Gaussian, $\alpha \in C_R(fg)$, i.e., $\alpha = \sum_{k=0}^{n+m}(\sum_{i+j=k} a_i b_j) r_k$ where r_k is an element of R for any $0 \leq k \leq nm$. Then, $\alpha = \phi(\alpha) = \sum_{k=0}^{n+m}(\sum_{i+j=k} a_i b_j)\phi(r_k)$ where ϕ is the module retraction map, which prove that $\alpha \in C_A(fg)$. Thus, A is a Gaussian ring.

2) Assume that (A, M) is a local ring and $R := A \propto (A/M)$ be the trivial ring extension of A by A/M. If R is Gaussian, then so is A by 1). Conversely, the fact that R is Gaussian in case A is Gaussian follows easily from the characterization of local Gaussian rings given by Tsang ([28]): a local ring R with maximal ideal M is Gaussian if and only if for any two elements a, b in M the following two conditions hold: 1) $(a, b)^2 = (a^2)$ or (b^2); 2) if $(a, b)^2 = (a^2)$ and $ab = 0$, then $b^2 = 0$. ■

The necessity of the conditions imposed in Theorem 2.1 will be proved in Examples 2.4 and 2.7.

Secondly, we study the transfer of Prüfer property between a ring and its subring retract. It is clear that each total ring of quotients is a Prüfer ring. Recall that an R-module E is called a torsion-free if for every regular element $a \in R$ and $e \in E$ such that $ae = 0$, we have $a = 0$ or $e = 0$.

Theorem 2.2 *Let R be a ring and A a subring retract of R.*
1) *Assume that the module retraction map $\phi : R \longrightarrow A$ verifies $Ker(\phi)$ is torsion-free. If R is a Prüfer ring then so is A.*
2) *Assume that (A, M) is a local total ring of quotients, where M is its maximal ideal; and assume that the module retraction map ϕ verifies $Mker(\phi) = 0$ and $ker(\phi) \subseteq Nil(R)$. Then R is a total ring of quotients; in particular, R is Prüfer.*

Proof. 1) Assume that $Ker(\phi)$ is torsion-free, where $\phi : R \longrightarrow A$ is the module retraction map, and R is a Prüfer ring. Let $I = \sum_{i=1}^{n} a_i A$ be a finitely generated regular ideal of A and a be a regular element of I. We tent to prove that I is invertible. Let b be an element of R such that $ba = 0$, we have $\phi(b)a = 0$ so $\phi(b) = 0$ since a is a regular element of A, i.e. $b \in Ker(\phi)$. In the other hand, by setting $b = a' + v \in R$ where $a' \in A$ and $v \in Ker(\phi)$ (since $Ker(\phi)$ is a direct summand of R), we obtain that $0 = a'a + va$, and so $a'a = va = 0$, therefore $a' = 0$ and $v = 0$ as a is regular in A and $Ker(\phi)$ is torsion-free; which proves that a is a regular element of R and so the ideal $J = \sum_{i=1}^{n} a_i R$ is a finitely generated regular ideal of R. Hence, since R is Prüfer, J is invertible in R and so the polynomial $f(X) = \sum_{i=1}^{n} a_i X^i$ is Gaussian in R (since $J = C_R(f)$). Using the proof of Theorem 2.1(1), we find that $f(X)$ is Gaussian in A; hence, as $I(= C_A(f))$ is a regular ideal of A, it is invertible in A (by [3, Theorem 4.2(2)]). Thus, A is Prüfer.

2) Assume that (A, M) is a local total ring of quotients, where M is its maximal ideal; and assume that the module retraction map ϕ verifies $Mker(\phi) = 0$ and $ker(\phi) \subseteq Nil(R)$. Set $V = Ker(\phi)$.
In order to show that R is a total ring of quotients, we have to prove that each element $a + v$ of R is invertible or zero-divisor element. Indeed:
If $a \in M$, then a is a non invertible element of A; that's a is zero-divisor in A (since A is a total ring of quotients). Hence there exists b nonzero element of M such that $ab = 0$. Therefore, $b(a + v) = 0$ as $MV = 0$, which means that $a + v$ is a zero-divisor element in R.
If $a \notin M$, then a is invertible in A and so in R; hence, $a+v$ is invertible in R as sum of an invertible element and a nilpotent one.
Thus, R is a total ring of quotients. ■

In the following example we prove that the retraction is not sufficient to transfer the Prüfer property.

Example 2.3 *Let (A, M) be a non Prüfer local ring and E a nonzero A-module such that $ME = 0$. Let $R := A \propto E$ be the trivial ring extension of A by E. Then:*
1) *R is a total ring of quotients (since R is local with maximal ideal $M \propto E$ and $(M \propto E)(0, e) = 0$ for each $e \in E$). In particular, R is Prüfer.*
2) *A is a non Prüfer subring retract of R.*

In the next example we ensure the necessity of the conditions imposed in Theorems 2.1(2) and 2.2(2).

Example 2.4 *Let (V, M) be a rank-one discrete valuation domain such that $2 \in M$ (for instance, $V := \mathbb{Z}_{(2)}$). Then $R := V \propto V$ is not Prüfer. In particular, R is not Gaussian.*

Proof. It suffices to show that R is not Prüfer. Let $I := R(2, 0) + R(2, 1)$ be a finitely generated ideal of R. It is clear that I is regular (since $(2, 0)$ is regular). Since R is local, the 2-generated regular ideal I is invertible if and only if it is principal. Then, again since R is local, I is principal if and only if it is generated by one of the two generators and this is false, so the conclusion follows easily. ■

We study now the transfer of arithmetical property between a ring and its subring retract.
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Theorem 2.5 *Let R be a ring and A a subring retract of R. If R is an arithmetical ring then so is A.*

Proof. By [17, Theorem 2], it suffices to show that for any pair of ideals I and J of A such that $I \subseteq J$ and J is finitely generated, there should exist an ideal H of A for which $I = HJ$.
We have $IR \subseteq JR$ and JR is a finitely generated ideal of R; so, as R is arithmetical, there exists an ideal L of R such that $IR = LJR$ that is $IR = LJ$. Therefore, $I = \phi(IR) = \phi(LJ) = \phi(L)J$ and so A is arithmetical. ■

In the following example we prove that, under the same conditions as in Theorem 2.1(2), we can't transfer the arithmetical property from A to R.

Example 2.6 *Let (A, M) be a valuation domain which is not a field, where M is its maximal ideal. Set $R = A \propto (A/M)$ be the trivial ring extension of A by A/M. Then:*
1) *A is an arithmetical subring retract of the local ring R.*
2) *R is not arithmetical.*

Proof. 1) is clear. Also, we claim that R is not arithmetical. Let $I := R(a, 0) + R(0, e)$ be a finitely generated ideal of R, where a is any nonzero element of M and e is any nonzero element of A/M. Since R is local, I is principal if and only if it is generated by one of the two generators and this is false, so the conclusion follows easily. ■

The following example proves that the condition "A is a subring retract of R" can not be removed in the proof of Theorems 2.1(1) and 2.5.

Example 2.7 *Let K be a field, $K[X, Y]$ the polynomial ring where X and Y are two indeterminate elements, and let $Q(K[X])$ be the quotient field of $K[X]$. Then $Q(K[X])[Y]$ is a Prüfer domain containing the subring $K[X, Y]$ which isn't a Prüfer domain.*

3 Applications

In this section we give two applications to the results obtained in section 2. The first application is devoted to trivial ring extension $R := A \propto E$ of a ring A by an A-module E. Recall that A is a module retract of R, where the module retraction map ϕ is defined by $\phi(x, e) = x$ and $Ker(\phi) = 0 \propto E$.

Proposition 3.1 *Let A be a ring, E an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then:*
1) a) *Assume that $E(= Ker(\phi))$ is torsion-free. If R is a Prüfer ring then so is A.*
b) *Assume that (A, M) is a local ring, where M is its maximal ideal such that $ME = 0$. Then R is a total ring of quotients. In particular, R is a Prüfer ring.*
2) a) *If R is Gaussian then so is A.*
b) *Assume that $E := A/M$, where M is a maximal ideal of A. Then R is a Gaussian ring if and only if so is A.*
3) *If R is arithmetical then so is A.*
4) *$wdim(R) > 1$.*

Proof. Let's remark first that $R := A \propto E$, where (A, M) is a local ring and $ME = 0$, is a total ring of quotients (since R is a local ring with maximal ideal $M \propto E$ and $(M \propto E)(0, 1) = 0_R$). By section 2, it remains to show that $wdim(R) > 1$.
Let $f \in E - \{0\}$ and $J := R(0, f)(= 0 \propto (Af))$. Consider the exact sequence of R-modules:

$$0 \longrightarrow Ker(u) \longrightarrow R \xrightarrow{u} J \longrightarrow 0$$

where $u(a, e) = (a, e)(0, f) = (0, af)$. Hence, $Ker(u) = Ann(f) \propto E$. We claim that J is not flat. Deny. Then, by [26, Theorem 3.55], $J = J \cap Ker(u) = JKer(u) = (0 \propto Af)(Ann(f) \propto E) = 0 \propto (Ann(f)f) = 0$, a contradiction. Therefore, J is not flat and $wdim(R) > 1$. ∎

As shown below, Proposition 3.1 enrichies the literature with new examples of non-Gaussian Prüfer rings.

Example 3.2 *Let (A, M) be a non-Prüfer local domain, where M is its maximal ideal, and let E be a nonzero A-module such that $ME = 0$. Let $R := A \propto E$ be the trivial ring extension of A by E. Then:*
1) *R is Prüfer by Proposition 3.1(1.b).*
2) *R is not Gaussian by Proposition 3.1(2.a) since A is not Gaussian (as A is non-Prüfer domain).*

For enrich the literature with new examples of non-arithmetical Gaussian rings, we propose the next two examples.

Example 3.3 *Let (A, M) be a valuation domain, where M is its maximal ideal, and let $R := A \propto (A/M)$ be the trivial ring extension of A by A/M. Then:*
1) *R is Gaussian by Theorem 3.1(2.b) since A is a valuation domain.*
2) *R is not arithmetical by Example 2.6(3).*

Example 3.4 *Let k be a proper subfield of a field K and let $R := k \propto K$ be the trivial ring extension of k by K. Then:*
1) *R is Gaussian by [2, Example 2.3(2.b)].*
2) *R is not arithmetical by [2, Example 2.3(2.c)] since R is local.*

Now we construct an arithmetical ring R such that $wdim(R) > 1$.

Example 3.5 *Let K be a field and $R := K \propto K$ be the trivial ring extension of K by K. Then:*
1) *R is arithmetical.*
2) *$wdim(R) = \infty$.*

Proof. 1) R is arithmetical by [2, Example 2.3(1.a)].
2) The ideal $I : R(0,1)$ is not flat by the proof of Proposition 3.1(4). On the other hand, the exact sequence of R-modules:

$$0 \longrightarrow I \longrightarrow R \xrightarrow{u} I \longrightarrow 0$$

where $u(a,e) = (a,e)(0,1) = (0,a)$ shows that $fd_R(I) = \infty$. Hence, $wdim(R) = \infty$ and this completes the proof. ■

Let's see in the following example that even if we replace the field K, in the above example, by a principal total ring of quotients A, we don't have in general $R := A \propto A$ is Gaussian; in particular it's not arithmetical.
The same example is a Prüfer non Gaussian ring.

Example 3.6 *Let $A := \mathbb{Z}/(2^i\mathbb{Z})$, where $i \geq 2$ be an integer, and let $R := A \propto A$ be the trivial ring extension of A by A. Then:*
1) *A is a local principal total ring of quotients with maximal ideal $M = 2A$.*
2) *R is a local total ring of quotients. In particular, R is a Prüfer ring.*
3) *R is not Gaussian. In particular, R is not arithmetical.*

Proof. 1) and **2)** are clear since R is local with maximal ideal $M \propto A$ and $(M \propto A)(0, 2^{i-1}) = 0_R$. It remains to show that R is not Gaussian. For that let $f := (2^{i-1}, 0) + (2^{i-1}, 1)X \in R[X]$. We have $f^2 = 0$ (and so $C_R(f^2) = 0$) and $(C_R(f))^2 = R(0, 2^{i-1})(\neq 0_R)$. Therefore, R is not Gaussian and this completes the proof of Example 3.6. ■

The second application is devoted to the Nagata rings. Let A be a ring and $R := A(X) = S^{-1}A[X]$ the localization of $A[X]$ by S, where S is the multiplicative set in $A[X]$ formed by all polynomials $f(X)$ such that $C(f) = A$. By construction we have $A(X) = A + XA[X] + C$ where $C = \{\frac{f(x)}{g(x)} / f(x), g(x) \in A[X], d^o f(X) < d^o g(x) \text{ and } C(g(x)) = A\}$ (by [21, Chapter IV, Proposition 1.4(1)]), which implies that A is a module retract of $A(X)$. The ring $R := A(X)$ is called the Nagata ring. See for instance [16, 21].

By section 2, we obtain:

Proposition 3.7 *Let A be a ring, $R := A(X)$ be the Nagata ring. Then:*
1) *Assume that $E(= Ker(\phi))$ is torsion-free. If R is a Prüfer ring then so is A.*
2) *If $R := A(X)$ is Gaussian then so is A.*
3) *If $R := A(X)$ is arithmetical then so is A.*

Recall that a subset S of a ring R is called dense if $Ann(S) = 0$. A ring R is called strongly Prüfer if every finitely generated dense ideal is locally principal. Notice that the Nagata ring $A(X)$ is Prüfer if and only if A is strongly Prüfer by [16, Theorem 18.10]. For instance, a strongly Prüfer ring is a Prüfer ring by [16, Lemme 18.1].

Recall that a ring R satisfies (CH)-property if each finitely generated ideal of R has a non-zero annihilator. It is clear that a (CH)-ring is strongly Prüfer ring. For instance, the trivial ring extension $R := A \propto E$ is a (CH)-ring (and so strongly Prüfer ring) for each local ring (A, M) (where M is its maximal ideal) and an A-module E such that $ME = 0$.

Now we construct a new examples of non-Gaussian Prüfer rings, as shown below.

Example 3.8 *Let A be a non-Gaussian (CH)-ring and let $R := A(X)$ be the Nagata ring. Then:*
1) *R is Prüfer.*
2) *R is a not Gaussian.*

Proof. 1) It is clear that A is strongly Prüfer ring since A is a (CH)-ring. Therefore R is a Prüfer ring by [16, Theorem 18.10].
2) R is not Gaussian by Proposition 3.6(2) since A is not Gaussian. ■

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